

## FAST DECREASING RATIONAL FUNCTIONS

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### ABSTRACT

Necessary and sufficient conditions are obtained for the existence of sequences of rational functions of the form  $r_n(x) = p_n(x)/p_n(-x)$ , with  $p_n$  a polynomial of degree  $n$ , that decrease geometrically on  $(0, 1]$  in accordance with a specified rate function. The technique of proof involves minimum energy problems for Green potentials in the presence of an external field. Applications are given for the construction of rational approximations of  $|x|$  and  $\operatorname{sgn}(x)$  on  $[-1, 1]$  having geometric rates of convergence for  $x \neq 0$ .

### 1. Introduction

Rational functions of the form

$$(1.1) \quad r_n(x) = P_n(x)/P_n(-x)$$

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( $P_n$  stands for a polynomial of degree  $\leq n$ ) play an important role in rational approximation on  $[-1, 1]$ . Once we have constructed  $r_n$  of the form (1.1) that is fast decreasing on  $[0, 1]$  (note that  $r_n(0) = 1$ ), we automatically get good rational approximants for  $|x|, \operatorname{sgn} x$ , and related functions (cf. [11], [4], [19], [8]). On the other hand, Newman's inequality (cf. [11], Lemma 3.1)

$$(1.2) \quad \int_x^1 \log \left| \frac{\zeta - t}{\zeta + t} \right| \frac{dt}{t} \geq -\frac{\pi^2}{2}, \quad 0 < x < 1, \quad \zeta \in \mathbb{C},$$

implies that any  $r_n$  of the form (1.1) must satisfy

$$(1.3) \quad \max_{[x,1]} |r_n| \geq \exp \left\{ -\pi^2 n / 2 \log \frac{1}{x} \right\}, \quad 0 < x < 1,$$

so that such  $r_n$  cannot decrease "too fast" (see also [3]). Let us formulate a general problem.

Given a function  $\varphi$  that is continuous on  $[0, 1]$  and satisfies  $\varphi(0) = 0$ , we wish to find a sequence  $r_n, n \geq 1$ , of functions of the form (1.1), with the property:

$$(1.4) \quad |r_n(x)| \leq C e^{-n\varphi(x)}, \quad 0 \leq x \leq 1, \quad n \geq 1.$$

Under what conditions on  $\varphi$  do there exist such  $r_n$ ? We also consider a weaker property, namely

$$(1.5) \quad |r_n(x)| \leq e^{o(n) - n\varphi(x)}, \quad 0 \leq x \leq 1, \quad n \geq 1,$$

where  $o(n)$  is uniform in  $x$ . Still weaker, is the property:

$$(1.6) \quad |r_n(x)| \leq C e^{-cn\varphi(x)}, \quad 0 \leq x \leq 1, \quad n \geq 1$$

for some positive constants  $C, c$ .

The existence of  $r_n$  satisfying (1.6) is settled by

**THEOREM 1.1:** *Let  $\varphi \in C[0, 1], \varphi(0) = 0$ . Then there exist  $r_n$  with property (1.6) if and only if*

$$(1.7) \quad \int_0^1 \frac{\varphi(x)}{x} dx < \infty.$$

This result was proved by V. Maimeskul and the authors in [8]. (Although not stated explicitly, it follows from Lemma 3.1 and Theorem 4.2 in [8].) The proof also provides an estimate for a constant  $c$  in the exponential term of (1.6). For example, if  $\varphi$  is increasing on  $[0, 1]$ , one can set

$$c = \left\{ \sum_{k=0}^{\infty} \varphi(2^{-k}) \right\}^{-1}.$$

However, the method of [8] is not suitable for producing sharp estimates.

In this paper we utilize a potential theoretic approach that yields best possible estimates. The method is similar to one used by V. Totik in his paper [16] on fast decreasing polynomials, and it is based on a study of a certain equilibrium problem. In our case, the problem is the following one.

Given a measure  $\mu \geq 0$  in  $D := \{z \in \mathbf{C} : \operatorname{Re} z > 0\}$ , let

$$(1.8) \quad U_g^\mu(z) := \int \log \left| \frac{z + \bar{t}}{z - t} \right| d\mu(t), \quad z \in D$$

denote the Green potential of  $\mu$ . Find  $\mu$  of total mass  $\|\mu\| = 1$ , with support  $S(\mu) \subseteq [0, 1]$ , that satisfies, with some constant  $c_\mu$ :

$$(1.9) \quad U_g^\mu - \varphi = c_\mu \quad \text{on } S(\mu),$$

$$(1.10) \quad U_g^\mu - \varphi \geq c_\mu \quad \text{on } [0, 1].$$

Note that this is a non-standard “singular” problem, since  $[0, 1]$  touches the boundary of  $D$ . Another complication is that the condition (1.7) allows  $\varphi$  of the form

$$\varphi(x) = \left( \log \frac{c}{x} \right)^{-\alpha}, \quad \alpha > 1, \quad c > 1.$$

Such a  $\varphi$  has an integrable derivative but  $\varphi'$  is not in  $L_p[0, 1]$ , for any  $p > 1$ . Therefore, known results concerning the density of  $\mu$  (cf. [9], [16]) are not applicable.

The relevance of the above equilibrium problem is obvious: given a polynomial

$$P_n(z) = \prod_{j=1}^n (z - \zeta_j),$$

we may write

$$\left| P_n(z)/P_n(-\bar{z}) \right| = \exp(-nU_g^{\nu_n}(z)),$$

where

$$\nu_n := \frac{1}{n} \sum_{j=1}^n \delta_{\zeta_j}, \quad \|\nu_n\| = 1,$$

and  $\delta_{\zeta_j}$  denotes the unit point mass at  $\zeta_j$ . Then the property (1.5) can be formulated as

$$(1.11) \quad U_g^{\nu_n}(x) - \varphi(x) \geq o(1), \quad x \in [0, 1]$$

while (1.4) becomes

$$(1.12) \quad U_g^{\nu_n}(x) - \varphi(x) \geq O(n^{-1}), \quad x \in [0, 1].$$

Thus, our goal is to describe  $\varphi$  for which  $c_\mu$  of (1.9) is zero; for then, on discretizing  $\mu$ , we may hope to get  $r_n$  of the form (1.1) that satisfy (1.4) or (1.5).

The paper is organized as follows. In the next section we collect some auxiliary results. In particular, a generalization of Mhaskar-Saff's *F-functional* is given. In Section 3 we show that the problem (1.9), (1.10) has a unique solution, and then we prove

**THEOREM 1.2:** *Given  $\varphi \in C[0, 1], \varphi(0) = 0$ , let  $U_g^\mu, c_\mu$  be as in (1.9). Then the following statements are equivalent:*

- (i) *There exist  $r_n, n \geq 1$ , satisfying (1.5);*
- (ii) *there holds*

$$(1.13) \quad \frac{2}{\pi^2} \int_0^1 (U_g^\mu(x) - c_\mu) \frac{dx}{x\sqrt{1-x^2}} \leq 1;$$

- (iii)  $c_\mu = 0$ .

Of course, it is desirable to have a condition in terms of  $\varphi$ . The necessary condition is immediate—if we have (1.5), then  $c_\mu = 0$  by (iii), so that  $\varphi \leq U_g^\mu$  by (1.10). Then (1.13) implies that

$$(1.14) \quad \frac{2}{\pi^2} \int_0^1 \varphi(x) \frac{dx}{x\sqrt{1-x^2}} \leq 1.$$

In many important cases this condition is also sufficient. In Section 4 we prove (compare with Theorem 3.3 in [16]) the following.

**THEOREM 1.3:** *Let  $\varphi \in C[0, 1], \varphi(0) = 0$ , and assume additionally that  $\varphi$  is increasing and concave on  $[0, 1]$ . Then there exist  $r_n, n \geq 1$ , with property (1.5) if and only if (1.14) holds.*

In Section 5 we consider the stronger property (1.4). Here we need to analyze the density of the equilibrium measure  $\mu$ . In doing so we impose extra conditions on  $\varphi$  that ensure a regular behavior of  $\mu'$  near 0. Assume that for  $x$  small enough,

- (a)  $x\varphi'(x)$  is increasing;
- (b) for some  $\delta > 0$ ,  $x\varphi'(x) \leq (1 - \delta)\varphi(x)$ .

Then we have

THEOREM 1.4: *Let  $\varphi$  be as in Theorem 1.3, and assume that  $\varphi$  satisfies (a), (b) for  $x$  small enough. Then there exist  $r_n, n \geq 1$ , with property (1.4) provided strict inequality holds in (1.14), i.e.*

$$(1.14') \quad \frac{2}{\pi^2} \int_0^1 \varphi(x) \frac{dx}{x\sqrt{1-x^2}} < 1.$$

*Remark:* Many important functions, like  $x^\alpha, \alpha < 1$ , or  $(\log c/x)^{-\alpha}, \alpha > 1$ , satisfy (a) and (b) above. Moreover, applying a finer discretization technique (cf. [7], [17]), one can replace (1.14') by (1.14), at least for the case  $\varphi(x) = cx^\alpha, \alpha < 1$ .

Finally, we apply our results to rational approximation of  $|x|$ . It is well-known that the error in best approximation of  $|x|$  by rational functions of order  $n$  behaves like  $\exp(-\pi\sqrt{n})$  (cf. [19],[14]). However, the best approximants do not converge to  $|x|$  geometrically fast on any subinterval of  $[-1, 1]$  (cf. [12]). In [8] near best rational approximants  $R_n$  were constructed with the property

$$(1.15) \quad ||x| - R_n(x)| \leq C \exp\{-c_1\sqrt{n} - c_2n\varphi(|x|)\}, \quad x \in [-1, 1].$$

The same method shows that  $c_1$  can be chosen arbitrarily close to  $\pi$ , but no sharp estimate of  $c_2$  was given. Here we prove a more precise result.

THEOREM 1.5: *Let  $\varphi$  be a continuous increasing function on  $[0, 1]$  with  $\varphi(0) = 0$ . Let  $0 \leq \varepsilon \leq 1$ , and assume that a sequence  $\{R_n\}, n \geq 1$ , exists such that for  $x \in [-1, 1]$*

$$(1.16) \quad ||x| - R_n(x)| \leq C \exp\left\{-\pi\sqrt{n(1-\varepsilon)} - n\varphi(|x|)\right\}.$$

*Then*

$$(1.17) \quad \frac{2}{\pi^2} \int_0^1 \varphi(x) \frac{dx}{x\sqrt{1-x^2}} \leq \varepsilon.$$

*In particular, if  $\varepsilon = 0$ , then  $\varphi \equiv 0$  so that geometric convergence is not possible.*

*Conversely, if strict inequality holds in (1.17) and  $\varphi$  is as in Theorem 1.4, then there exist  $\{R_n\}$  satisfying (1.16).*

## 2. Auxiliary results

We start with a simple observation. Let  $E$  be a compact set in the open right half-plane  $D$ , and let  $\tilde{E} := \{-\bar{z} : z \in E\}$  be its reflection about the  $y$ -axis. The

pair  $(E, \tilde{E})$  is called a (symmetric) **condenser**. Given a positive measure  $\mu$  on  $E$ , define  $\tilde{\mu}$  on  $\tilde{E}$ , by symmetry. Then the Green potential of  $\mu$  can be written as an ordinary logarithmic potential of the signed measure  $\mu - \tilde{\mu}$  :

$$U_g^\mu(z) = \int \log \left| \frac{z + \bar{t}}{z - t} \right| d\mu(t) = \int \log \frac{1}{|z - t|} d(\mu - \tilde{\mu})(t).$$

With this observation in mind, the classical result of T. Bagby [2] can be stated as follows (see also [20] or [13, Section II.5]).

There exists a unique positive measure  $\omega_E$  on  $E$ , of total mass  $|\omega_E| = 1$ , such that

$$(2.1) \quad U_g^{\omega_E} = \frac{1}{\text{cap}_g E} := \frac{1}{2\text{cap}(E, \tilde{E})}, \quad \text{q.e. on } E$$

and

$$(2.2) \quad U_g^{\omega_E} \leq \frac{1}{\text{cap}_g E}, \quad \text{in } \bar{\mathbf{C}}.$$

(As usual, q.e. means “except for a set of zero logarithmic capacity.”) The measure  $\omega_E$  is called the **equilibrium measure** (or distribution) on  $E$  relative to  $D$ , and the constant  $\text{cap}_g E$  is called the **Green capacity** of  $E$  relative to  $D$  (it also coincides with the capacity of the condenser formed by  $E$  and the  $y$ -axis).

In particular, for  $E = [\varepsilon, 1], 0 < \varepsilon < 1$ , we have (cf. [1])

$$(2.3) \quad \frac{1}{\text{cap}_g E} = \frac{\pi K(\varepsilon)}{K'(\varepsilon)},$$

where  $K(\varepsilon), K'(\varepsilon)$  denote the complete elliptic integrals for moduli  $\varepsilon, \sqrt{1 - \varepsilon^2}$ , respectively. The equilibrium measure  $\omega_{[\varepsilon, 1]} =: \omega_E$  is given by

$$(2.4) \quad d\omega_E(t) = \frac{1}{K'(\varepsilon)} \{(1 - t^2)(t^2 - \varepsilon^2)\}^{-1/2} dt, \quad t \in [\varepsilon, 1].$$

Next, we shall need a maximum principle for Green potentials (cf. [13, Section II.5]). We state it for the special case  $S(\mu) \subseteq [0, 1]$ , since this will suffice for our purposes, and the proof in this case is elementary. Indeed, it is easy to see that

$$(2.5) \quad \left| \frac{z + t}{z - t} \right| < \left| \frac{\text{Re } z + t}{\text{Re } z - t} \right|, \quad \text{Re } z > 0, \quad \text{Im } z \neq 0, \quad t > 0,$$

and

$$(2.6) \quad \left| \frac{z + t}{z - t} \right| < \frac{1 + t}{1 - t}, \quad \text{Re } z > 1, \quad 0 < t < 1.$$

Therefore,  $U_g^\mu$  attains its maximum in  $D$  only on  $[0, 1]$ . Since  $U_g^\mu$  is convex on any subinterval of  $[0, +\infty) \setminus S(\mu)$ , we obtain that

$$(2.7) \quad U_g^\mu(z) \leq \sup_{S(\mu)} U_g^\mu, \quad z \in D, \quad S(\mu) \subseteq [0, 1].$$

Moreover, the maximum principle for harmonic functions shows that strict inequality holds for  $z \notin S(\mu)$ , unless  $S(\mu) = \{0\}$ , in which case  $U_g^\mu \equiv 0$ .

We are now in a position to prove an important result that can be viewed as a strengthened version of Newman's inequality (1.2).

LEMMA 2.1: *Let  $\sigma$  be the measure on  $(0, 1)$  given by*

$$(2.8) \quad d\sigma(t) = \frac{2}{\pi^2} \frac{dt}{t\sqrt{1-t^2}}, \quad t \in (0, 1)$$

(note that  $\|\sigma\| = \infty$ ). Then

(i)

$$(2.9) \quad U_g^\sigma(z) = 1, \quad z \in (0, 1],$$

$$(2.10) \quad U_g^\sigma(z) < 1, \quad z \in D \setminus (0, 1].$$

Furthermore,

(ii) *for any positive finite measure  $\mu$  in  $D$  with  $S(\mu) \subset \overline{D}$ , we have*

$$(2.11) \quad \int U_g^\mu d\sigma \leq \|\mu\|$$

and

$$(2.12) \quad \int U_g^\mu d\sigma = \|\mu\| - \mu\{0\}, \quad \text{if } S(\mu) \subseteq [0, 1].$$

*Proof:* (i) For  $0 \leq \varepsilon < 1$ , set

$$I_\varepsilon(x) := \int_\varepsilon^1 \log \left| \frac{x+t}{x-t} \right| \{ (1-t^2)(t^2 - \varepsilon^2) \}^{-1/2} dt.$$

Then (see (2.1), (2.3), (2.4) ) for  $\varepsilon > 0$ , we have

$$I_\varepsilon(x) = \pi K(\varepsilon), \quad \varepsilon \leq x \leq 1.$$

Since  $K(\varepsilon) \rightarrow \pi/2$  as  $\varepsilon \rightarrow 0$  (cf. [1]), equation (2.9) will follow provided we can show that  $I_\varepsilon$  converges to  $I_0$ . Fix  $0 < x \leq 1$  and write, for  $\varepsilon < x/2$ ,

$$I_\varepsilon = \int_\varepsilon^{x/2} + \int_{x/2}^1 =: I_{\varepsilon 1} + I_{\varepsilon 2}.$$

The integrand in  $I_{\varepsilon 2}$  is decreasing as  $\varepsilon \rightarrow 0$ , so the monotone convergence theorem applies. In  $I_{\varepsilon 1}$ , we make a substitution  $t^2 = x^2 s/4 + \varepsilon^2(1 - s)$ . Then

$$I_{\varepsilon 1} = \sqrt{(x^2/4) - \varepsilon^2} \int_0^1 \psi(t) \frac{ds}{\sqrt{s}},$$

where

$$\psi(t) := \frac{1}{2t} \log \left| \frac{x+t}{x-t} \right| \frac{1}{\sqrt{1-t^2}}, \quad t^2 = \frac{x^2}{4} s + \varepsilon^2(1 - s).$$

Since  $\psi(t)$  is bounded on  $(0, x/2]$ , we may appeal to the bounded convergence theorem. Consequently, (2.9) holds and then (2.10) follows by the maximum principle.

(ii) By the Fubini–Tonelli theorem,

$$\int U_g^\mu d\sigma = \int U_g^\sigma d\mu,$$

so that (2.11), (2.12) follow from (2.9) and (2.10). ■

We turn now to the equilibrium problem in the presence of an external field. Given a continuous real-valued function  $\varphi$  on a compact set  $E \subset D$ , with  $\text{cap } E > 0$ , then it is known (cf. [13, Section II.5]) that there exists a unique measure  $\mu$  on  $E$ , having total mass 1 and such that, with some constant  $c_\mu$ ,

$$(2.13) \quad U_g^\mu - \varphi \leq c_\mu \quad \text{on } S(\mu),$$

$$(2.14) \quad U_g^\mu - \varphi \geq c_\mu \quad \text{q.e. on } E.$$

Now, let  $K$  be a compact subset of  $E$  (of positive capacity). Integrating (2.14) against the equilibrium measure  $\omega_K$ , we obtain (recall (2.1)):

$$c_\mu \leq \int U_g^\mu d\omega_K - \int \varphi d\omega_K = \int U_g^{\omega_K} d\mu - \int \varphi d\omega_K \leq \frac{1}{\text{cap}_g K} - \int \varphi d\omega_K.$$

For  $K = S(\mu)$  the opposite inequality holds (integrate (2.13) against  $\omega_K$  and apply (2.1), recalling that sets of zero capacity have zero  $\omega_K$ -measure). We have thus proved the following.

**THEOREM 2.2:** *Define, for any compact  $K \subset E$ ,*

$$(2.15) \quad F(K) := -\frac{1}{\text{cap}_g K} + \int \varphi d\omega_K.$$

Then

$$\max_K F(K) = F(S(\mu)) = -c_\mu.$$

*Remark:* The  $F$ -functional

$$F(K) = \log(\text{cap } K) - \int Q \, d\omega_K$$

(note:  $\text{cap}$  not  $\text{cap}_g$ ) was introduced and studied by Mhaskar and Saff [10] in connection with the equilibrium problem for logarithmic potentials of positive measures. Theorem 2.2 is an extension of their result. (Our  $\varphi$  is  $-Q$  in the notation of [10].) Note that this can be extended further to the non-symmetric case.

We conclude with two observations. Let  $E$  be a regular compact set (e.g., a segment) in  $D$ , let  $\nu$  be any measure in  $D$ , and assume that, for some continuous  $\varphi$  on  $E$ , the relation  $U_g^\nu - \varphi \geq c$  holds q.e. on  $E$ . Then it must hold everywhere on  $E$  (cf. [16, p.138]).

Next, since

$$U_g^\nu(z) = \int \log \frac{1}{|z-t|} d\nu(t) + u(z),$$

where  $u$  is harmonic in  $D$ , the principle of descent and the lower envelope theorem (cf. [15, Appendix], [6], or [18]) can be applied to  $U_g^\nu$ .

### 3. The equilibrium problem

Let  $E_\varepsilon := [\varepsilon, 1]$ . We have seen in Section 2 that there exists unique  $\mu_\varepsilon$  on  $[\varepsilon, 1]$ ,  $\|\mu_\varepsilon\| = 1$ , such that

$$(3.1) \quad U_g^{\mu_\varepsilon} - \varphi = c_\varepsilon, \quad \text{on } S(\mu_\varepsilon),$$

$$(3.2) \quad U_g^{\mu_\varepsilon} - \varphi \geq c_\varepsilon, \quad \text{on } [\varepsilon, 1],$$

where  $c_\varepsilon$  is some constant. We now show that, as  $\varepsilon \rightarrow 0$ ,  $\mu_\varepsilon$  approaches the desired equilibrium measure  $\mu$ , that satisfies (1.9), (1.10). Integrating (3.2) against  $dt/t$  we obtain

$$\left\{ c_\varepsilon + \min_{[\varepsilon, \delta]} \varphi \right\} \log \frac{\delta}{\varepsilon} \leq \int_\varepsilon^\delta U_g^{\mu_\varepsilon}(t) \frac{dt}{t} = \int \left\{ \int_\varepsilon^\delta \log \left| \frac{x+t}{x-t} \right| \frac{dt}{t} \right\} d\mu_\varepsilon(x) \leq \frac{\pi^2}{2}.$$

In the last step we used Newman's inequality (1.2) and the fact that  $\|\mu_\varepsilon\| = 1$ . Since  $\varphi$  is continuous and  $\varphi(0) = 0$ , we obtain, on letting first  $\varepsilon \rightarrow 0$  and then  $\delta \rightarrow 0$ , that

$$\limsup_{\varepsilon \rightarrow 0} c_\varepsilon \leq 0.$$

On the other hand, as  $U_g^{\mu_\epsilon} \geq 0$ , inequality (3.1) shows that  $c_\epsilon$  is bounded from below, by  $-\max_{[0,1]} \varphi$ .

Now, let  $\mu$  be any weak-star limit of  $\{\mu_\epsilon\}$ . Passing to a subsequence we may assume that  $c_\epsilon \rightarrow c_\mu \leq 0$ .

Applying (in  $D$ ) the principle of descent to (3.1) and the lower envelope theorem to (3.2), we obtain that

$$U_g^\mu - \varphi \leq c_\mu \quad \text{on } S(\mu) \setminus \{0\},$$

$$U_g^\mu - \varphi \geq c_\mu \quad \text{q.e. on } [0, 1].$$

The second relation holds (see the end of Section 2) everywhere on  $(0, 1]$ , and as  $c_\mu \leq 0$  it also holds at 0. The first relation holds at 0 if 0 is an isolated point of  $S(\mu)$ , since then  $U_g^\mu$  must be continuous at 0. In the opposite case we have

$$c_\mu \geq \liminf_{\substack{x \rightarrow 0 \\ x \in S(\mu)}} (U_g^\mu - \varphi) \geq 0.$$

Thus,  $c_\mu = 0$  and again  $U_g^\mu - \varphi = c_\mu$  at  $0 \in S(\mu)$ . Therefore we finally obtain

$$(3.3) \quad U_g^\mu - \varphi = c_\mu, \quad \text{on } S(\mu),$$

$$(3.4) \quad U_g^\mu - \varphi \geq c_\mu, \quad \text{on } [0, 1].$$

Being a weak-star limit of unit measures,  $\mu$  is also unit, and the existence of the solution of problem (1.9), (1.10) is established.

*Remarks:* (a) It may happen above that  $\mu$  has a mass at 0. For example, if  $\varphi \equiv 0$ , then  $c_\mu$  must be 0 (since  $U_g^\mu \geq 0$ ,  $c_\mu \leq 0$ ), and then (3.3) yields  $U_g^\mu = 0$  on  $S(\mu)$ . By the maximum principle,  $U_g^\mu \leq 0$  in  $D$ , so that  $U_g^\mu$  is identically 0. Therefore, the corresponding  $\mu$  is a unit mass placed at the origin.

(b) By (3.3),  $U_g^\mu$  is continuous on  $S(\mu)$ ; hence ([18, Theorem III.2]) it is continuous in  $\mathbf{C}$ , except perhaps at 0. Now, if 0 is not a limit point of  $S(\mu)$ , this potential is obviously continuous at 0. In the opposite case,  $c_\mu = 0$  and (3.3) shows that  $\lim_{x \rightarrow 0^+} U_g^\mu(x) = 0$ . Then we have  $\lim_{z \rightarrow 0, z \in D} U_g^\mu(z) = 0$  by the same argument that was used in the proof of the maximum principle (2.7). Finally, as  $U_g^\mu(-z) = U_g^\mu(z)$ , we obtain that  $U_g^\mu$  is continuous in  $\mathbf{C}$ .

Next, we prove the uniqueness of  $\mu$  above. This follows from a more general result that shows that  $U_g^\mu - c_\mu$  is the least superharmonic majorant for  $\varphi$ , in the class of all majorants of the form  $U_g^\nu + \text{const}$ ,  $\|\nu\| \leq 1$ .

LEMMA 3.1: Let  $\mu$  satisfy (3.3), (3.4),  $\|\mu\| = 1$ . Assume that  $U_g^\nu - a \geq \varphi$  on  $[0, 1]$ , where  $\|\nu\| \leq 1$  and  $a$  is a constant (necessarily, non-positive). Then

$$(3.5) \quad \varphi \leq U_g^\mu - c_\mu \leq U_g^\nu - a \quad \text{on } [0, 1].$$

*Proof:* We consider two cases.

CASE I:  $a \leq c_\mu$ .

Let  $h = U_g^\nu - U_g^\mu$ .  $h$  is superharmonic in  $D \setminus S(\mu)$ , bounded from below ( $U_g^\nu \geq 0$  while  $U_g^\mu$  is bounded) and satisfies

$$\liminf_{z \rightarrow \partial D} h \geq 0.$$

On  $S(\mu)$  we have

$$h = U_g^\nu - U_g^\mu = (U_g^\nu - \varphi) - (U_g^\mu - \varphi) \geq a - c_\mu.$$

Now, since  $U_g^\mu$  is continuous,  $h$  is lower semicontinuous. Hence,

$$\liminf_{z \rightarrow x \in S(\mu)} h(z) \geq h(x) \geq a - c_\mu.$$

Applying the minimum principle for superharmonic functions, we conclude (as  $a - c_\mu \leq 0$ ) that  $h \geq a - c_\mu$  in  $D$ , and (3.5) follows.

CASE II:  $a > c_\mu$ .

Then the same reasoning as above gives  $h \geq 0$  in  $D$ . Next, since  $a \leq 0$ , we see that  $c_\mu < 0$ , so that  $0 \notin S(\mu)$ . By Lemma 2.1 (apply (2.12) to  $\mu$  and (2.11) to  $\nu$ ) we obtain, as  $h \geq 0$  and  $\|\nu\| \leq \|\mu\|$  :

$$0 \geq \|\nu\| - \|\mu\| \geq \int (U_g^\nu - U_g^\mu) d\sigma = \int h d\sigma \geq \int_{S(\mu)} h d\sigma \geq (a - c_\mu) \int_{S(\mu)} d\sigma > 0$$

and we have a contradiction. Thus, Case II is impossible, and the lemma is proved. ■

We now proceed with the

*Proof of Theorem 1.2:* (i)  $\Rightarrow$  (ii) Assume (1.5) holds. This means (see (1.11)) that with  $\|\nu_n\| \leq 1$ , the relation  $U_g^{\nu_n}(x) - \varphi(x) \geq o(1)$  holds uniformly for  $x \in [0, 1]$ . Let  $\nu$  be any weak-star limit of  $\nu_n$ . Then  $U_g^\nu \geq \varphi$  on  $[0, 1]$ , and Lemma 3.1 yields  $U_g^\mu - c_\mu \leq U_g^\nu$ . Integrating this against  $d\sigma$  (see Lemma 2.1 (ii)), we obtain (1.13).

(ii)  $\Rightarrow$  (iii) This is obvious since otherwise the integral in (1.13) diverges.

(iii)  $\Rightarrow$  (i). Here we know that  $\varphi \leq U_g^\mu$  on  $[0, 1]$ ,  $U_g^\mu$  is continuous,  $S(\mu) \subseteq [0, 1]$  and  $\|\mu\| = 1$ . We need to construct  $\nu_n$ ,  $\|\nu_n\| \leq 1$ , such that (1.11) will hold. We follow the reasoning in [9, pp. 40-43]. Assume first that  $\mu\{0\} = 0$ , and define  $0 = t_0 < t_1 < \dots < t_n = 1$  by

$$(3.6) \quad \int_{t_{k-1}}^{t_k} d\mu = \frac{1}{n}, \quad k = 1, 2, \dots, n.$$

Let  $\nu_n$  be a measure having a mass  $1/n$  at each  $t_j$ ,  $1 \leq j \leq n$ . Fix  $x \in [0, 1]$  and let  $t_{j-1} < x \leq t_j$ , for some  $j$ . Since  $\log|(x+t)/(x-t)|$  is increasing on  $[0, t_{j-1}]$  and is decreasing on  $[t_j, 1]$ , a simple estimation gives

$$(3.7) \quad U_g^{\nu_n}(x) \geq U_g^\mu(x) - \int_{t_{j-1}}^{t_j} \log \left| \frac{x+t}{x-t} \right| d\mu(t).$$

If  $t \in [t_{j-1}, t_j]$  satisfies  $|t-x| > n^{-1}$ , we have

$$\log \left| \frac{x+t}{x-t} \right| < \log(1+2n),$$

so that the integration over such  $t$ 's will contribute  $O(\log n/n)$  to the integral in (3.7). If we can show that

$$(3.8) \quad \int_{|t-x| \leq n^{-1}} \log \left| \frac{x+t}{x-t} \right| d\mu(t) = o(1), \quad n \rightarrow \infty$$

uniformly for  $x \in [0, 1]$ , we are done because (3.7) then gives

$$U_g^{\nu_n} \geq U_g^\mu - o(1) \geq \varphi - o(1), \quad \text{on } [0, 1]$$

as required.

Assuming (3.8) is false, one can find  $x \in [0, 1]$  and a sequence  $x_n \rightarrow x$  such that

$$(3.9) \quad \int_{|t-x_n| \leq n^{-1}} \log \left| \frac{x_n+t}{x_n-t} \right| d\mu(t) \geq \alpha > 0, \quad n \geq 1.$$

Now let  $\delta > 0$ . Since  $x_n \rightarrow x$  and the integrand in (3.9) is nonnegative, we only strengthen (3.9) if we replace the range of integration by  $|t-x| < \delta$ , provided  $n$  is large enough, i.e.

$$(3.10) \quad \int_{|t-x| < \delta} \log \left| \frac{x_n+t}{x_n-t} \right| d\mu(t) \geq \alpha, \quad n \geq n_\delta.$$

Next, as  $\mu$  has no point masses, we can select  $\delta$  so that

$$\int_{|t-x|\geq\delta} \log \left| \frac{x+t}{x-t} \right| d\mu(t) \geq U_g^\mu(x) - \frac{\alpha}{2}.$$

For  $n$  large enough, we can replace  $x$  by  $x_n$  in the integrand, thereby changing the integral by  $o(1)$ . Then, on adding the resulting inequality to (3.10) we obtain

$$U_g^\mu(x_n) \geq U_g^\mu(x) + \frac{\alpha}{2} - o(1), \quad n \geq n_\delta,$$

contradicting the continuity of  $U_g^\mu$ .

Finally, if  $\mu\{0\} := 1 - \rho > 0$ , define  $\mu_\rho$  on  $(0, 1]$  by  $\mu_\rho := \mu/\rho$ . Then we obtain a unit measure and, by the preceding argument, we have a stronger result with  $\varphi$  replaced by  $U_g^{\mu_\rho}$ . ■

#### 4. Proof of Theorem 1.3

Let

$$(4.1) \quad \rho := \frac{2}{\pi^2} \int_0^1 \varphi(x) \frac{dx}{x\sqrt{1-x^2}}.$$

Recall that the condition  $\rho \leq 1$  is necessary for (1.5) to hold. Thus, in view of Theorem 1.2, we only need to show that if  $\rho \leq 1$  and  $\varphi$  is increasing and concave, then  $c_\mu = 0$ . We will prove a stronger result, part (ii) of which will be used in Section 5.

**THEOREM 4.1:** *Let  $\varphi, \rho$  be as above, and assume that  $0 < \rho \leq 1$ . Then*

(i) *the equilibrium potential  $U_g^\mu$  satisfies*

$$(4.2) \quad U_g^\mu = \varphi, \quad \text{on } [0, 1].$$

*Moreover,  $S(\mu) = [0, 1]$  and  $\mu$  has a mass  $1 - \rho$  at 0.*

(ii) *Let  $\tilde{\mu}$  denote the restriction of  $\mu$  on  $(0, 1]$  (so that (4.2) holds with  $\mu$  replaced by  $\tilde{\mu}$ ). Then  $\tilde{\mu}$  is absolutely continuous with respect to  $dt$ , and its density is given by*

$$(4.3) \quad v(t) = \frac{2}{\pi^2} t(1-t^2)^{-1/2} PV \int_0^1 \frac{\varphi'(s)}{t^2 - s^2} (1-s^2)^{1/2} ds,$$

*for a.e.  $t \in [0, 1]$ .*

*Proof:* (i) Proceeding as at the beginning of Section 3, we get a unit measure  $\mu_\epsilon$  on  $[\epsilon, 1]$  that satisfies

$$(4.4) \quad U_g^{\mu_\epsilon} - \varphi = c_\epsilon, \quad \text{on } S(\mu_\epsilon),$$

$$(4.5) \quad U_g^{\mu_\epsilon} - \varphi \geq c_\epsilon, \quad \text{on } [\epsilon, 1].$$

Recall that we have shown that  $\lim_{\epsilon \rightarrow 0} c_\epsilon = c_\mu \leq 0$ .

Now, let  $I$  be an interval in the complement  $[\epsilon, 1] \setminus S(\mu_\epsilon)$  with endpoints belonging to  $S(\mu_\epsilon)$ . Then  $U_g^{\mu_\epsilon}$  is strictly convex on  $I$  while  $\varphi$  is concave. Since (4.4) holds at the endpoints of  $I$ , (4.5) is violated inside  $I$ . Thus, there is no such  $I$ , which means that  $S(\mu_\epsilon)$  is an interval, say  $[a_\epsilon, b_\epsilon]$ . Next, (4.4) holds at  $b_\epsilon$ , and  $U_g^{\mu_\epsilon}$  is decreasing for  $x > b_\epsilon$ . Since  $\varphi$  is increasing,  $b_\epsilon$  must be equal to 1, since otherwise (4.5) is false on  $(b_\epsilon, 1]$ . So,  $S(\mu_\epsilon) = [a_\epsilon, 1]$ . Now, by Theorem 2.2,  $c_\epsilon$  is given by

$$-c_\epsilon = F([a_\epsilon, 1]) = \frac{1}{K'(a_\epsilon)} \left\{ \int_{a_\epsilon}^1 \varphi(t) \{(1-t^2)(t^2-\epsilon^2)\}^{-1/2} dt - \pi K(a_\epsilon) \right\},$$

where we used (2.3), (2.4).

Let  $I_\epsilon$  denote the integral in  $\{ \}$ . Putting  $t^2 = s + a_\epsilon^2(1-s)$  we obtain that

$$I_\epsilon = \int_0^1 \psi(s + a_\epsilon^2(1-s)) \frac{ds}{\sqrt{s(1-s)}}, \quad \psi(t) := \frac{\varphi(\sqrt{t})}{2\sqrt{t}}.$$

Since  $\varphi(0) = 0$  and  $\varphi$  is concave, we have  $\sqrt{t} \varphi'(\sqrt{t}) \leq \varphi(\sqrt{t})$ , which means that  $\psi(t)$  is decreasing. Therefore,

$$I_\epsilon \leq \int_0^1 \psi(s) \frac{ds}{\sqrt{s(1-s)}} = \int_0^1 \varphi(t) \frac{dt}{t\sqrt{1-t^2}} \leq \frac{\pi^2}{2},$$

by our assumption  $\rho \leq 1$ . On the other hand,  $\pi K(a_\epsilon) > \pi^2/2$ . Thus,  $c_\epsilon > 0$  and we have

$$0 \geq c_\mu = \lim_{\epsilon \rightarrow 0} c_\epsilon \geq 0.$$

Hence,  $c_\mu = 0$ , so that  $U_g^\mu - \varphi = 0$  on  $S(\mu)$ . Since  $U_g^\mu - \varphi = 0$  at 0 as well, the same reasoning used for  $\mu_\epsilon$  shows that  $S(\mu) = [0, 1]$ . Thus, (4.2) holds. Recalling the definition of  $\rho$  we obtain from Lemma 2.1(ii):

$$\rho = \int \varphi d\sigma = \int U_g^\mu d\sigma = \|\mu\| - \mu(\{0\}) = 1 - \mu(\{0\}).$$

(ii) We use the method of [9], but care must be taken near 0, since  $\varphi'$  may not be in  $L_p[0, 1], p > 1$ . So let us consider

$$(4.6) \quad \varphi_\epsilon(x) := \begin{cases} \varphi(x), & x \in [\epsilon, 1] \\ x \frac{\varphi(\epsilon)}{\epsilon}, & x \in [0, \epsilon], \end{cases}$$

which is bounded. Next, extend  $\varphi_\varepsilon$  to  $[-1, 0]$  as an *odd* function. Then we may apply the known result (cf. [9] or [13, Ch. IV.3]), which asserts that the function

$$(4.7) \quad v_\varepsilon(t) := \frac{1}{\pi^2} \sqrt{\frac{1+t}{1-t}} \text{PV} \int_{-1}^1 \sqrt{\frac{1-s}{1+s}} \frac{\varphi'_\varepsilon(s)}{t-s} ds + \frac{A}{\sqrt{1-t^2}}$$

is defined a.e. on  $[-1, 1]$ , belongs to  $L_p[-1, 1]$ ,  $1 < p < 2$ , and satisfies (with any choice of a constant  $A$ ):

$$(4.8) \quad \int_{-1}^1 \log \frac{1}{|x-t|} v_\varepsilon(t) dt = \varphi_\varepsilon(x) + c_A, \quad x \in [-1, 1].$$

*Remark:* It is assumed in the above references that the function  $\varphi_\varepsilon$  ( $-f$ , in their notation) is *even*, while our  $\varphi_\varepsilon$  is *odd*. However, this assumption was used for purposes other than proving (4.7), (4.8).

Since we want to return to Green potentials, we choose  $A$  to ensure that  $v_\varepsilon$  is odd. Set

$$A := \frac{1}{\pi^2} \text{PV} \int_{-1}^1 \sqrt{\frac{1-s}{1+s}} \frac{\varphi'_\varepsilon(s)}{s} ds$$

(since  $\varphi'_\varepsilon = \text{const}$  near 0, the PV-integral exists). With the above choice of  $A$ , we have

$$v_\varepsilon(t) = \frac{1}{\pi^2 \sqrt{1-t^2}} \text{PV} \int_{-1}^1 \sqrt{\frac{1-s}{1+s}} \varphi'_\varepsilon(s) \left( \frac{1+t}{t-s} + \frac{1}{s} \right) ds,$$

that is,

$$(4.9) \quad \begin{aligned} v_\varepsilon(t) &= \frac{t}{\pi^2 \sqrt{1-t^2}} \text{PV} \int_{-1}^1 \sqrt{1-s^2} \frac{\varphi'_\varepsilon(s)}{t-s} \frac{ds}{s} \\ &= \frac{2t}{\pi^2 \sqrt{1-t^2}} \text{PV} \int_0^1 \sqrt{1-s^2} \frac{\varphi'_\varepsilon(s)}{t^2-s^2} ds, \end{aligned}$$

where, in the last step, we used the evenness of  $\varphi'_\varepsilon$ . Thus,  $v_\varepsilon$  is odd. Then the integral in (4.8) is also odd in  $x$ , and so is  $\varphi_\varepsilon$ . Hence,  $c_A = 0$  and applying the oddness of  $v_\varepsilon$ , we may rewrite (4.8) as

$$(4.10) \quad \int_0^1 \log \left| \frac{x+t}{x-t} \right| v_\varepsilon(t) dt = \varphi_\varepsilon(x), \quad x \in [0, 1].$$

Next, we verify that  $v_\varepsilon > 0$  on  $(0, 1)$ . A standard calculation shows that

$$(4.11) \quad \text{PV} \int_0^1 \sqrt{1-s^2} \frac{ds}{t^2-s^2} = \frac{\pi}{2}, \quad t \in (-1, 1) \setminus \{0\}.$$

Therefore  $v_\varepsilon$  can be written (use the second relation in (4.9)) as:

$$(4.12) \quad v_\varepsilon(t) = \frac{1}{\pi} \frac{t\varphi'_\varepsilon(t)}{\sqrt{1-t^2}} + \frac{2t}{\pi^2\sqrt{1-t^2}} \int_0^1 \sqrt{1-s^2} \frac{\varphi'_\varepsilon(s) - \varphi'_\varepsilon(t)}{t^2 - s^2} ds.$$

Since  $\varphi'_\varepsilon$  is positive and nonincreasing on  $[0, 1]$ , both terms in (4.12) are positive; hence  $v_\varepsilon > 0$ .

Now, define  $\mu_\varepsilon \geq 0$  on  $[0, 1]$  by  $d\mu_\varepsilon(t) := v_\varepsilon(t)dt$ . (The present  $\mu_\varepsilon$  should not be confused with the one that appeared before.) Then (4.10) and Lemma 2.1(ii) yield

$$\|\mu_\varepsilon\| = \int \varphi_\varepsilon d\sigma.$$

Let  $\varepsilon \rightarrow 0$ , and let  $\tilde{\mu}$  be any weak-star limit of  $\mu_\varepsilon$ . The usual reasoning gives (as  $\varphi_\varepsilon \rightarrow \varphi$ ) that  $U\tilde{\mu} = \varphi$  on  $[0, 1]$ . Since  $\varphi_\varepsilon \uparrow \varphi$ , we obtain

$$(4.13) \quad \|\tilde{\mu}\| = \lim_{\varepsilon \rightarrow 0} \|\mu_\varepsilon\| = \int \varphi d\sigma = \rho$$

(see (4.1)). The uniqueness of the equilibrium measure then shows that  $\tilde{\mu}$  coincides with  $\tilde{\mu}$  as defined in Theorem 4.1 (the restriction of  $\mu$  on  $(0, 1]$ ). To complete the proof, it remains to show that  $v$ , as given by (4.3), is the limit of  $v_\varepsilon$ , in  $L_1[0, 1]$ . Note that the existence a.e of the integral (4.3) follows from the established existence of the corresponding integral (4.9), and as  $\varphi_\varepsilon = \varphi$  on  $[\varepsilon, 1]$ .

Exactly as we deduced (4.12) from (4.9), we may write  $v$  in the form

$$(4.14) \quad v(t) = \frac{1}{\pi} \frac{t\varphi'(t)}{\sqrt{1-t^2}} + \frac{2t}{\pi^2\sqrt{1-t^2}} \int_0^1 \sqrt{1-s^2} \frac{\varphi'(s) - \varphi'(t)}{t^2 - s^2} ds.$$

Now, fix  $0 < \delta < 1/2$  and rewrite (4.12), for  $t \in [\delta, 1 - \delta]$  and  $\varepsilon < \frac{1}{2}\delta$ , in the form

$$v_\varepsilon(t) = \frac{t\varphi'(t)}{\pi\sqrt{1-t^2}} + \frac{2t}{\pi^2\sqrt{1-t^2}} \left( \int_\varepsilon^1 \sqrt{1-s^2} \frac{\varphi'(s) - \varphi'(t)}{t^2 - s^2} ds + \int_0^\varepsilon \frac{\varepsilon^{-1}\varphi(\varepsilon) - \varphi'(t)}{t^2 - s^2} \sqrt{1-s^2} ds \right),$$

where we used the definition (4.6) of  $\varphi_\varepsilon$ .

As  $\varepsilon \rightarrow 0$ , the second integral is  $O(\varphi(\varepsilon)) \rightarrow 0$ , uniformly for  $t \in [\delta, 1 - \delta]$ . The first integral is increasing to the integral in (4.14). Since  $\int_0^1 v_\varepsilon = \|\mu_\varepsilon\|$  are uniformly bounded (see (4.13)) we conclude that  $v \in L_1[\delta, 1 - \delta]$ , that

$$(4.15) \quad \int_\delta^{1-\delta} v = \lim_{\varepsilon \rightarrow 0} \int_\delta^{1-\delta} v_\varepsilon \leq C,$$

and that

$$(4.16) \quad \int_{\delta}^{1-\delta} |v - v_{\varepsilon}| \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

Since (4.15) holds for any  $\delta > 0$ , we obtain that  $v \in L_1[0, 1]$ . Therefore,

$$\left( \int_0^{\delta} + \int_{1-\delta}^1 \right) v \rightarrow 0, \quad \delta \rightarrow 0.$$

The same is true for  $v_{\varepsilon}$ , uniformly on  $\varepsilon$ , as (4.13) shows. Combined with (4.16) this gives:  $v_{\varepsilon} \rightarrow v$  in  $L_1[0, 1]$ , and the proof is complete. ■

**5. Proof of Theorem 1.4**

We first show that the extra conditions (a), (b) of Theorem 1.4 ensure that, for some  $C_1, C_2 > 0$ ,

$$(5.1) \quad C_1 \leq v(t) t/\varphi(t) \leq C_2, \quad t \text{ small,}$$

where  $v$  is defined in (4.3). Extend  $\varphi$  to  $[-1, 1]$  as an odd function and rewrite (4.3), using the evenness of  $\varphi'$ :

$$v(t) = \frac{t}{\pi^2 \sqrt{1-t^2}} PV \int_{-1}^1 \sqrt{1-s^2} \frac{\varphi'(s)}{t-s} \frac{ds}{s}.$$

With the aid of

$$\frac{t}{(t-s)s} = \frac{1}{t-s} + \frac{1}{s}$$

we further rewrite  $v$  as follows:

$$v(t) = \frac{1}{\pi^2 \sqrt{1-t^2}} PV \int_{-1}^1 \sqrt{1-s^2} \frac{\varphi'(s)}{t-s} ds,$$

since the integral involving  $1/s$  vanishes because the integrand is an odd function.

Next, write

$$\sqrt{1-s^2} = \sqrt{1-t^2} + (\sqrt{1-s^2} - \sqrt{1-t^2}).$$

Then we obtain

$$(5.2) \quad v(t) = \frac{1}{\pi^2} PV \int_{-1}^1 \frac{\varphi'(s)}{t-s} ds + \frac{B(t)}{\sqrt{1-t^2}},$$

where

$$0 < B(t) < ct, \quad t \in (0, 1].$$

Now express the integral in (5.2) as

$$\int_{-1}^1 \frac{\varphi'(s) - \varphi'(t)}{t - s} + \varphi'(t) \log \frac{1+t}{1-t}.$$

The second term is positive. Also,

$$\int_{-1}^{-t} + \int_t^1 = 2t \int_t^1 \frac{\varphi'(s) - \varphi'(t)}{t^2 - s^2} ds > 0$$

as  $\varphi'$  is decreasing. Furthermore,

$$\begin{aligned} \int_{-t}^t &= 2t \int_0^t \frac{\varphi'(s) - \varphi'(t)}{t^2 - s^2} ds > \int_0^t \frac{\varphi'(s) - \varphi'(t)}{t - s} ds \\ &> \frac{1}{t} \int_0^t [\varphi'(s) - \varphi'(t)] ds = \frac{\varphi(t)}{t} - \varphi'(t) \geq \delta \frac{\varphi(t)}{t} \end{aligned}$$

by the assumption (b), provided  $t$  is small. Collecting all estimates, we get the left-hand inequality in (5.1).

Now we return to the integral in (5.2). We have

$$\left( \int_{-1}^{-2t} + \int_{2t}^1 \right) \frac{\varphi'(s)}{t - s} ds = 2t \int_{2t}^1 \frac{\varphi'(s)}{t^2 - s^2} ds < 0.$$

Also,

$$\int_{-2t}^0 \frac{\varphi'(s)}{t - s} ds < \frac{1}{t} \int_{-2t}^0 \varphi'(s) ds = \frac{\varphi(2t)}{t} < \frac{2\varphi(t)}{t},$$

by concavity. Finally,

$$\begin{aligned} PV \int_0^{2t} \frac{\varphi'(s)}{t - s} ds &= \int_0^{2t} \frac{\varphi'(s) - \varphi'(t)}{t - s} ds = \frac{1}{t} \int_0^{2t} \frac{(t - s + s)\varphi'(s) - t\varphi'(t)}{t - s} ds \\ &= \frac{1}{t} \varphi(2t) + \frac{1}{t} \int_0^{2t} \frac{s\varphi'(s) - t\varphi'(t)}{t - s} ds < \frac{1}{t} \varphi(2t), \end{aligned}$$

since the integrand is negative, by the assumption (a). Thus we have the second inequality in (5.1).

We can now proceed with the

*Proof of Theorem 1.4:* We have to construct a measure  $\nu_n$  having a mass  $1/n$  at some  $t_1, \dots, t_n$  and such that (see (1.12))

$$U_g^{\nu_n} \geq \varphi - \frac{c}{n}, \quad \text{on } [0, 1].$$

Consider the function  $(1 + \varepsilon)\varphi$ . Since strict inequality holds in (1.14'), there is  $\varepsilon > 0$  such that

$$\frac{2}{\pi^2} \int_0^1 (1 + \varepsilon)\varphi(x) \frac{dx}{x\sqrt{1-x^2}} = 1.$$

By Theorem 4.1, the equilibrium measure  $\mu$  for  $(1 + \varepsilon)\varphi$  has no mass at 0 and it satisfies  $U_g^\mu = (1 + \varepsilon)\varphi$  on  $[0, 1]$ . Let  $0 = t_0 < t_1 < \dots < t_n = 1$  and  $\nu_n$  be as in the proof of Theorem 1.2. The relation (3.7) now becomes

$$U_g^{\nu_n}(x) \geq (1 + \varepsilon)\varphi(x) - \int_{t_{j-1}}^{t_j} \log \left| \frac{x+t}{x-t} \right| d\mu(t), \quad x \in (t_{j-1}, t_j).$$

Thus, our task is to show that

$$(5.3) \quad \int_{t_{j-1}}^{t_j} \log \left| \frac{x+t}{x-t} \right| d\mu(t) \leq \varepsilon\varphi(x) + \frac{c}{n}$$

for some  $C$  and for all  $x \in (t_{j-1}, t_j)$ ,  $j = 1, \dots, n$ . We have already shown that the above integral is  $o(1)$  uniformly in  $x \in (t_{j-1}, t_j)$  and in  $j$ . Therefore, (5.3) is obvious for  $x \geq \delta$  (any  $\delta > 0$ ) provided  $n \geq n(\delta)$ , and we may only consider the case  $t_{j-1} < x < t_j < \delta$ , where (a), (b) hold.

By the lower bound in (5.1) and by concavity of  $\varphi$ ,

$$\frac{1}{n} = \int_{t_{j-1}}^{t_j} v(t)dt \geq \frac{1}{c_1} \int_{t_{j-1}}^{t_j} \frac{\varphi(t)}{t} dt \geq \frac{1}{c_1} \int_{t_{j-1}}^{t_j} \varphi'(t)dt = \frac{1}{c_1}(\varphi(t_j) - \varphi(t_{j-1})).$$

Therefore, for  $t_{j-1} < x < t_j$ ,

$$(5.4) \quad \max_{[t_{j-1}, t_j]} \varphi = \varphi(t_j) \leq \varphi(x) + \frac{c_1}{n},$$

uniformly in  $j$ , provided  $t_j \leq \delta$ . Next,

$$\int_{t_{j-1}}^{t_j} \log \left| \frac{x+t}{x-t} \right| v(t)dt \leq \left\{ \int_{t_{j-1}}^{t_j} \log^2 \left| \frac{x+t}{x-t} \right| \frac{dt}{t} \right\}^{\frac{1}{2}} \left\{ \int_{t_{j-1}}^{t_j} tv^2(t)dt \right\}^{\frac{1}{2}}.$$

The first integral is bounded by an absolute constant (put  $t = \tau x$ ). The second is bounded by

$$\begin{aligned} \left\{ \int_{t_{j-1}}^{t_j} v(t)dt \right\}^{\frac{1}{2}} \max_{[t_{j-1}, t_j]} \{tv(t)\}^{\frac{1}{2}} &\leq c_2 n^{-1/2} \max_{[t_{j-1}, t_j]} \varphi(t)^{1/2} \\ &\leq c_3 \left[ n^{-1/2} \varphi(x)^{1/2} + n^{-1} \right], \end{aligned}$$

where we used the definition of  $t_j$ 's, the upper bound in (5.1) and, in the last step, (5.4). Since for any  $\varepsilon > 0$ ,

$$c_3 \sqrt{\frac{\varphi}{n}} \leq \varepsilon \varphi + \frac{c_3^2}{4\varepsilon n},$$

the bound (5.3) follows. ■

### 6. Rational approximation of $\operatorname{sgn} x$ and $|x|$

Throughout this section,  $r_n$  denotes a function of the form (1.1), while  $R_n$  denotes some rational function of order  $n$ , not necessarily of the form (1.1).

We start with a lemma that is due to D. Newman [11], except for a minor modification.

LEMMA 6.1: *Let  $\delta_n$  denote a positive function on  $[0,1]$ .*

(i) *Assume that*

$$(6.1) \quad \left| |x| - R_n(x) \right| \leq \delta_n(|x|), \quad x \in [-1, 1].$$

*Then there exists  $r_n$  that satisfies*

$$(6.2) \quad x|r_n(x)| \leq 2\delta_n(x), \quad x \in [0, 1].$$

*Moreover, we may assume that all zeros of  $r_n$  lie on  $(0,1]$ .*

(ii) *Assume that*

$$(6.3) \quad \left| \operatorname{sgn} x - R_n(x) \right| \leq \delta_n(|x|), \quad x \in [-1, 1].$$

*Then there exists  $r_n$  such that*

$$(6.4) \quad |r_n(x)| \leq 2\delta_n(x), \quad x \in [0, 1].$$

(iii) *Assume that  $r_n$  satisfies (6.2) and, additionally,*

$$(6.5) \quad 1 + r_n(x) \geq \frac{1}{2}, \quad x \in [0, 1].$$

*Then there exists  $R_n$  that satisfies (6.1) with  $\delta_n$  replaced by  $4\delta_n$ . Similarly, if  $r_n$  satisfies (6.4) and (6.5), then there is  $R_n$  satisfying (6.3) with  $\delta_n$  replaced by  $4\delta_n$ .*

*Proof:* (i) If  $R_n$  is even, (6.2) follows from [11], with factor 2 dropped. For arbitrary  $R_n := p_n/q_n$ , we may assume  $q_n > 0$  on  $[-1,1]$ . Then (6.1) implies

$$\left| \frac{xq_n(\pm x) - p_n(\pm x)}{q_n(x) + q_n(-x)} \right| \leq \left| \frac{xq_n(\pm x) - p_n(\pm x)}{q_n(\pm x)} \right| \leq \delta_n(x),$$

for  $x \in [0, 1]$ . Therefore, with

$$\tilde{R}_n(x) := \frac{p_n(x) + p_n(-x)}{q_n(x) + q_n(-x)}$$

we have  $|x - \tilde{R}_n(x)| \leq 2\delta_n(x)$ ,  $x \in [0, 1]$ . Since  $\tilde{R}_n$  is even, we may continue as in [11].

Next, let  $|(x - \xi)/(x + \xi)|$  be one of the factors of  $|r_n|$ . In view of (2.5), (2.6) we only strengthen (6.2) if we replace  $\xi \in D$  by a suitable  $\tilde{\xi} \in (0, 1]$ . For  $\xi \notin D$ , the above factor is  $\geq 1$  and we may drop it.

(ii) This follows easily from (i).

(iii) See [11]. ■

To ensure (6.5) for  $r_n$  satisfying (1.4), a small adjustment is needed. Let

$$|r_n(x)| \leq Ce^{-n\varphi(x)}, \quad x \in [0, 1].$$

Assuming  $\varphi$  is increasing, define  $0 < \alpha_n < 1$  by  $n\varphi(\alpha_n) = \log 2C$ . Now, if  $r_n$  has a zero,  $\xi$ , on  $(0, \alpha_n)$  we replace it by  $\alpha_n$ . Since

$$\left| \frac{x - \alpha_n}{x + \alpha_n} \right| < \left| \frac{x - \xi}{x + \xi} \right|, \quad x > \alpha_n, \quad \xi < \alpha_n$$

we see that a new  $r_n$  still satisfies (1.4) on  $[\alpha_n, 1]$ . On  $[0, \alpha_n]$ ,  $|r_n| \leq 1$  while  $C \exp(-n\varphi(x)) \geq \frac{1}{2}$ . Thus (1.4) still holds with  $C$  replaced by  $2C$ . Therefore a new  $r_n$  satisfies (1.4) and also (6.5). With these preliminaries, the following result is an immediate consequence of the necessary condition (1.14) and Theorem 1.4.

**THEOREM 6.2:** *Let  $\varphi$  be continuous and increasing on  $[0, 1]$  with  $\varphi(0) = 0$ . Assume there exist  $R_n, n \geq 1$ , such that*

$$(6.6) \quad |\operatorname{sgn} x - R_n(x)| \leq C \exp(-n\varphi(x)), \quad x \in [-1, 1].$$

*Then  $\varphi$  satisfies (1.14).*

*Conversely, if strict inequality holds in (1.14) and  $\varphi$  is as in Theorem 1.4, then there exist  $R_n, n \geq 1$  satisfying (6.6).*

We now turn to the approximation of  $|x|$ .

*Proof of the second part of Theorem 1.5:* Given  $\varepsilon > 0$ , let  $n$  be large enough. Since  $\varphi$  satisfies (1.17) with strict inequality, (1.14') holds for  $\varepsilon^{-1}\varphi$ . Hence there is an  $r_{[n\varepsilon]}$  such that

$$|r_{[n\varepsilon]}| \leq ce^{-[n\varepsilon]\varepsilon^{-1}\varphi} \leq c_1 e^{-n\varphi}, \quad \text{on } [0, 1].$$

Next, put  $p := [n(1 - \varepsilon)]$ . Then (cf. [19]) there is an  $r_p$  such that  $1 + r_p \geq \frac{1}{2}$  on  $[0,1]$  and

$$x|r_p(x)| \leq c_2 e^{-\pi\sqrt{p}}.$$

Note that  $c_1, c_2$  are independent of  $n$ . Since  $|r_{[n\varepsilon]}| \leq 1$ , we have  $1 + r_p r_{[n\varepsilon]} \geq \frac{1}{2}$  on  $[0, 1]$ . Then Lemma 6.1(iii) yields  $R_n$  of order  $p + [n\varepsilon] \leq n$  that satisfies (1.16), with a constant independent of  $n$ . ■

*Proof of the first part of Theorem 1.5:* Assume (1.16) holds. Since  $\varphi$  is increasing,  $\varphi > 0$  on  $(0, 1]$ . Then, applying Lemma 6.1(i), we get  $r_n$  that satisfy

$$(6.7) \quad x|r_n(x)| \leq c \exp\left(-\pi\sqrt{n(1-\varepsilon)}\right), \quad x \in [0, 1]$$

and also

$$(6.8) \quad x|r_n(x)| \leq ce^{-n\varphi(x)}, \quad x \in [0, 1].$$

Fix  $\eta > 0$ , and let  $t_1, \dots, t_n$  be zeros of  $r_n$  on  $(0, 1]$ . We show below that

$$(6.9) \quad N := \#\{t_j : t_j > \frac{1}{n}\eta\} \leq \varepsilon n + O(\sqrt{n} \log n).$$

Note that for other zeros we have

$$\left| \frac{x - t_j}{x + t_j} \right| \geq 1 - \frac{2}{n}, \quad t_j \leq \frac{1}{n}\eta, \quad x \in [\eta, 1],$$

so that

$$\prod_{t_j \leq \frac{1}{n}\eta} \left| \frac{x - t_j}{x + t_j} \right| \geq e^{-2}, \quad x \in [\eta, 1].$$

Therefore, we obtain from (6.8)

$$|r_N(x)| \leq c\eta^{-1}e^2e^{-n\varphi(x)}, \quad x \in [\eta, 1],$$

where  $r_N$  is of degree  $N$ . On taking logarithms and integrating against  $d\sigma$  (see Lemma 2.1) we obtain

$$N = \int_0^1 \geq \int_\eta^1 \log |r_N^{-1}| d\sigma \geq c_1 \int_\eta^1 d\sigma + n \int_\eta^1 \varphi d\sigma.$$

Dividing by  $n$  and letting  $n \rightarrow \infty$  yields, by (6.9),

$$\varepsilon \geq \int_\eta^1 \varphi d\sigma.$$

Since this holds for any  $\eta > 0$ , (1.17) follows.

It remains to prove (6.9). The same reasoning as above shows that

$$\prod_{t_j > \frac{1}{n}\eta} \left| \frac{x - t_j}{x + t_j} \right| \geq e^{-2}, \quad x \in [0, n^{-2}\eta].$$

Thus, we get from (6.7) that

$$x|r_{n-N}(x)| \leq ce^2 \exp\left(-\pi\sqrt{n(1-\varepsilon)}\right), \quad x \in [0, n^{-2}\eta].$$

On taking logarithms, dividing by  $x$  and integrating from  $\varepsilon_n := \eta n^{-2} \exp(-\pi\sqrt{n-N})$  to  $\eta n^{-2}$ , we obtain first, by Newman's inequality, and then dividing by  $\int_{\varepsilon_n}^{\eta n^{-2}} dx/x$ , that

$$\frac{1}{2} \frac{\log^2 n^{-2}\eta - \log^2 \varepsilon_n}{\log n^{-2}\eta - \log \varepsilon_n} - \frac{\pi^2}{2} \frac{n-N}{\log n^{-2}\eta/\varepsilon_n} \leq c_1 - \pi\sqrt{n(1-\varepsilon)}.$$

With our choice of  $\varepsilon_n$ , we get

$$\log n^{-2}\eta - \pi\sqrt{n-N} \leq c_1 - \pi\sqrt{n(1-\varepsilon)}$$

and (6.9) follows. ■

## References

- [1] N. I. Akhiezer, *Elements of the Theory of Elliptic Functions*, Translations of the American Mathematical Society, Vol. 79, Providence, NJ, 1990.
- [2] T. Bagby, *The modulus of plane condenser*, Journal of Mathematical Mechanics **17** (1967), 315–329.
- [3] A. A. Gonchar, *Estimates of growth of rational functions and some of their applications*, Matematicheskii Sbornik **72** (1967), 489–503; English translation: Mathematics of the USSR-Sbornik **1** (1967), 445–456.
- [4] A. A. Gonchar, *On the speed of rational approximation of continuous functions with characteristic singularities*, Matematicheskii Sbornik **73** (1967), 630–678; English translation: Mathematics of the USSR-Sbornik **2** (1967), 561–568.
- [5] K. G. Ivanov and V. Totik, *Fast decreasing polynomials*, Constructive Approximation **6** (1990), 1–20.
- [6] N. S. Landkof, *Foundations of Modern Potential Theory*, Springer-Verlag, Berlin, 1972.
- [7] A. L. Levin and D. S. Lubinsky, *Orthogonal polynomials and Christoffel functions for  $\exp(-|x|^\alpha)$ ,  $\alpha < 1$* , Journal of Approximation Theory **80** (1995), 219–252.

- [8] A. L. Levin, V. V. Maimeskul and E. B. Saff, *Rational approximation with locally geometric rates*, Journal of Approximation Theory **92** (1998), 308–330.
- [9] D. S. Lubinsky and E. B. Saff, *Strong Asymptotics for Extremal Polynomials Associated with Weights on  $\mathbb{R}$* , Lecture Notes in Mathematics, Vol. 1305, Springer-Verlag, Berlin, 1988.
- [10] H. N. Mhaskar and E. B. Saff, *Where does the sup norm of a weighted polynomial live? (A generalization of incomplete polynomials)*, Constructive Approximation **1** (1985), 71–91.
- [11] D. J. Newman, *Rational approximation to  $|x|$* , The Michigan Mathematical Journal **11** (1964), 11–14.
- [12] E. B. Saff and H. Stahl, *Asymptotic distribution of poles and zeros of best rational approximants to  $x^\alpha$  on  $[0, 1]$* , in *Topics in Complex Analysis*, Banach Center Publications, Vol. 31, Institute of Mathematics, Polish Academy of Sciences, Warsaw, 1995, pp. 329–348.
- [13] E. B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, Springer-Verlag, Berlin, 1997.
- [14] H. Stahl, *Best uniform rational approximation of  $|x|$  on  $[-1, 1]$* , Matematicheskii Sbornik **183** (1992), 85–112; English translation: Russian Academy of Sciences. Sbornik Mathematics **76** (1993), 461–487.
- [15] H. Stahl and V. Totik, *General Orthogonal Polynomials*, Encyclopedia of Mathematics and its Applications, Vol. 43, Cambridge University Press, Cambridge, 1992.
- [16] V. Totik, *Fast decreasing polynomials via potentials*, Journal d'Analyse Mathématique **62** (1994), 131–154.
- [17] V. Totik, *Weighted Approximation with Varying Weights*, Lecture Notes in Mathematics, Vol. 1569, Springer-Verlag, Berlin, 1994.
- [18] M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen, Tokyo, 1959.
- [19] N. S. Vyacheslavov, *On the uniform approximation of  $|x|$  by rational functions*, Soviet Mathematics Doklady **16** (1975), 100–104.
- [20] H. Widom, *Rational approximation and  $n$ -dimensional diameter*, Journal of Approximation Theory **5** (1972), 343–361.